5. Each customer who enters a certain business must be served first by server 1, then by server 2, and finally by server 3. The amount of time it takes to be served by server i is an exponential random variable with rate μ_i $(i = 1, 2, 3)$. Suppose you enter the system at a time when it contains a single customer who is being served by server 3.

(2 marks) (a) What is the probability that server 3 will still be busy when you move over to server 2?

(2 marks) (b) What is the probability that server 3 will still be busy when you move over to server 3?

(6 marks) (c) Suppose that $\mu_1 = \mu_3 = 1$ and $\mu_2 = 2$. Find the expected time that you spend in the system.

5(a) $\dfrac{\mu_1}{\mu_1+\mu_3}$.

5(b) $P(X_1+X_2<X_3)=\dfrac{\mu_1+\mu_2}{\mu_1+\mu_2+\mu_3}$

5(c) $\mathbb{E}[T]=S_1+S_2+S_3+W_3$

$\boxed{\text{method 1}}$

$$=\frac{1}{\mu_1}+\frac{1}{\mu_2}+\frac{1}{\mu_3}+\frac{\mu_1+\mu_2}{\mu_1+\mu_2+\mu_3}\cdot\frac{1}{\mu_3}$$

$$=\frac{1}{1}+\frac{1}{2}+1+\frac{3}{4}\cdot\frac{1}{1}$$

$$=2+\frac{1}{2}+\frac{3}{4}$$

$$=2+\frac{2+3}{4}=2+\frac{5}{4}=\frac{8+5}{4}=\boxed{\frac{13}{4}}.$$

$\boxed{\text{method 2}}$:

$$\mathbb{E}[T]=T_1+R_1$$

$$=\frac{1}{\mu_1+\mu_3}+R_1$$

$$R_1=\frac{\mu_1}{\mu_1+\mu_3}T_2+\frac{\mu_3}{\mu_1+\mu_3}\left(\frac{1}{\mu_2}+\frac{1}{\mu_3}+\frac{1}{\mu_1}\right)$$

$$T_2=\frac{1}{\mu_2+\mu_3}+R_2=\frac{1}{3}+\frac{11}{6}=\frac{2+11}{6}=\frac{13}{6}$$

$$R_2=\frac{\mu_2}{\mu_2+\mu_3}\left(\frac{2}{\mu_3}\right)+\frac{\mu_3}{\mu_2+\mu_3}\left(\frac{1}{\mu_2}+\frac{1}{\mu_3}\right)=\frac{2}{3}(2)+\frac{1}{3}\left(\frac{1}{2}+1\right)$$

$$=\frac{4}{3}+\frac{1}{\cancel{3}}\left(\frac{\cancel{3}}{2}\right)=\frac{4}{3}+\frac{1}{2}$$

R_2

$$=\frac{8+3}{6}=\frac{11}{6}.$$

$$R_1 = \frac{1}{2}\left(\frac{13}{6}\right) + \frac{1}{2}\left(2+\frac{1}{2}\right)$$

$$= \frac{1}{2} \cdot \frac{13}{6} + \frac{1}{2}\left(\frac{5}{2}\right)$$

$$= \frac{13}{12} + \frac{5}{4}$$

$$= \frac{13+5\times 3}{12} = \frac{13+15}{12} = \frac{28}{12} = \frac{4\times 7}{4\times 3} = \frac{7}{3}$$

$$E[T] = \frac{1}{2} + \frac{7}{3} = \frac{3+14}{6} = \boxed{\frac{17}{6}}$$