Coordinates of foci:

$$\left(\pm\sqrt{a^2+b^2},0\right)$$

Slope of m_1 of line PF_1:

$$m_1=\frac{y_0}{x_0+\sqrt{a^2+b^2}}$$

Slope of m_2 $of\ line\ PF_2$:

$$m_2=\frac{y_0}{x_0-\sqrt{a^2+b^2}}$$

So, the tangent line is:

$$y=m\left(x-x_0\right)+y_0$$

By substituting for y in the equation of the hyperbola you get:

$$\frac{x^2}{a^2}-\frac{\left(m\left(x-x_0\right)+y_0\right)^2}{b^2}=1$$

You know have a quadratic in x which you can work to arrange in standard form:

$$b^2x^2-a^2\left(m\left(x-x_0\right)+y_0\right)^2=a^2b^2$$

$$b^2x^2-a^2\left(m^2(x-x_0)^2+2my_0\left(x-x_0\right)+y_0^2\right)=a^2b^2$$

$$b^2x^2-a^2\left(m^2\left(x^2-2x_0x+x_0^2\right)+2my_0\left(x-x_0\right)+y_0^2\right)=a^2b^2$$

$$b^2x^2-a^2\left(m^2x^2-2m^2x_0x+m^2x_0^2+2my_0x-2my_0x_0+y_0^2\right)=a^2b^2$$

$$b^2x^2-a^2m^2x^2+2a^2m^2x_0x-a^2m^2x_0^2-2a^2my_0x+2a^2my_0x_0-a^2y_0^2=a^2b^2$$

$$\left(b^2-a^2m^2\right)x^2+2a^2m\left(mx_0-y_0\right)x-a^2\left(\left(mx_0-y_0\right)^2+b^2\right)=0$$

Now, having arranged the quadratic in standard form, we can equate the discriminant to zero, to find the value of mm, since the tangent line does not touch the hyperbola anywhere else:

$$\left(2a^2m\left(mx_0-y_0\right)\right)^2-4\left(b^2-a^2m^2\right)\left(-a^2\left(\left(mx_0-y_0\right)^2+b^2\right)\right)=0$$

Which reduces to:

$$\left(a^2 - x_0^2\right) m^2 + 2x_0 y_0 m - \left(b^2 + y_0^2\right) = 0$$

Applying the quadratic formula, you get:

$$m = \frac{-2x_0 y_0 \pm \sqrt{(2x_0 y_0)^2 + 4\left(a^2 - x_0^2\right)\left(b^2 + y_0^2\right)}}{2\left(a^2 - x_0^2\right)}$$

$$m = \frac{-x_0 y_0 \pm \sqrt{a^2 b^2 + a^2 y_0^2 - b^2 x_0^2}}{a^2 - x_0^2}$$

Now, since the point (x_0, y_0) is on the hyperbola, we know:

$$\frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} = 1$$

And this is arranged as:

$$a^2 b^2 + a^2 y_0^2 - b^2 x_0^2 = 0$$

And:

$$a^2 b^2 + a^2 y_0^2 - b^2 x_0^2 = 0$$

Using:

$$a^2 b^2 + a^2 y_0^2 - b^2 x_0^2 = 0$$

We find:

$$x_0^2 - a^2 = \frac{a^2 y_0^2}{b^2}$$

And so:

$$m = \frac{x_0 y_0}{\frac{a^2 y_0^2}{b^2}} = \frac{b^2 x_0}{a^2 y_0}$$

Now for calculus:

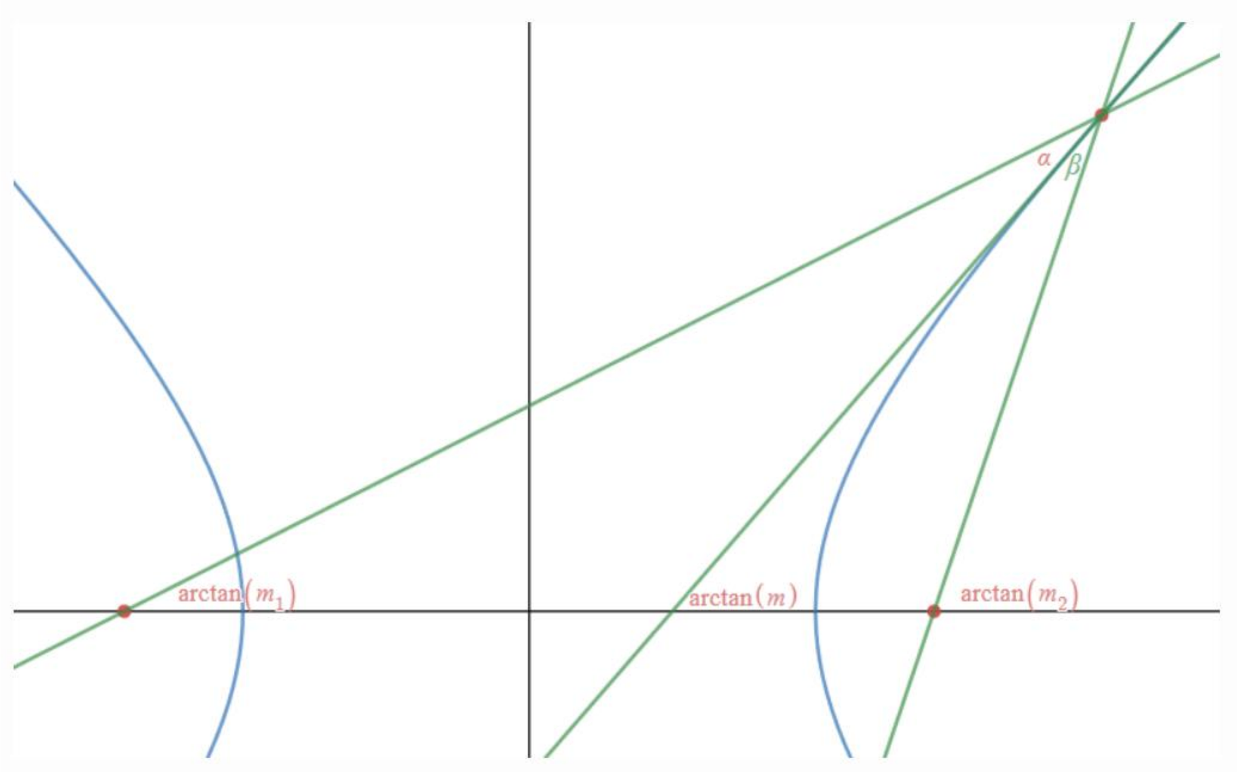
Implicitly differentiating the hyperbola with respect to x, we find:

$$\frac{2x}{a^2} - \frac{2y}{b^2} y' = 0 \implies y' = \frac{b^2 x}{a^2 y}$$

And so, at point (x_0, y_0), we get:

$$y' = \frac{b^2 x_0}{a^2 y_0} \quad \checkmark$$

So now considering this diagram:



We know:

$$\arctan(m_1) + (\pi - \arctan(m)) + \alpha = \pi \implies \alpha = \arctan(m) - \arctan(m_1)$$

$$\arctan(m) + (\pi - \arctan(m_2)) + \beta = \pi \implies \beta = \arctan(m_2) - \arctan(m)$$

Now considering:

$$\arctan(a) - \arctan(b) = \arctan\left(\frac{a-b}{1+ab}\right)$$

And:

$$\alpha = \arctan\left(\frac{m - m_1}{1 + mm_1}\right) = \arctan\left(\frac{\frac{b^2 x_0}{a^2 y_0} - \frac{y_0}{x_0 + \sqrt{a^2 + b^2}}}{1 + \frac{b^2 x_0}{a^2 y_0} \cdot \frac{y_0}{x_0 + \sqrt{a^2 + b^2}}}\right)$$

$$\beta = \arctan\left(\frac{m_2 - m}{1 + mm_2}\right) = \arctan\left(\frac{\frac{y_0}{x_0 - \sqrt{a^2 + b^2}} - \frac{b^2 x_0}{a^2 y_0}}{1 + \frac{b^2 x_0}{a^2 y_0} \cdot \frac{y_0}{x_0 - \sqrt{a^2 + b^2}}}\right)$$

And to simplify:

$$\frac{\frac{b^2 x_0}{a^2 y_0} - \frac{y_0}{x_0 + \sqrt{a^2 + b^2}}}{1 + \frac{b^2 x_0}{a^2 y_0} \cdot \frac{y_0}{x_0 + \sqrt{a^2 + b^2}}} = \frac{b^2 x_0^2 + b^2 x_0 \sqrt{a^2 + b^2} - a^2 y_0^2}{a^2 x_0 y_0 + a^2 y_0 \sqrt{a^2 + b^2} + b^2 x_0 y_0} = \frac{a^2 b^2 + b^2 x_0 \sqrt{a^2 + b^2}}{\left(a^2 + b^2\right) x_0 y_0 + a^2 y_0 \sqrt{a^2 + b^2}} = \frac{b^2 \left(a^2 + x_0 \sqrt{a^2 + b^2}\right)}{y_0 \sqrt{a^2 + b^2} \left(\sqrt{a^2 + b^2} x_0 + a^2\right)}$$

$$= \frac{b^2}{y_0 \sqrt{a^2 + b^2}}$$

Likewise:

$$\frac{\frac{y_0}{x_0 - \sqrt{a^2 + b^2}} - \frac{b^2 x_0}{a^2 y_0}}{1 + \frac{b^2 x_0}{a^2 y_0} \cdot \frac{y_0}{x_0 - \sqrt{a^2 + b^2}}} = \frac{b^2}{y_0 \sqrt{a^2 + b^2}}$$

And so, we can conclude:

$$\alpha = \beta$$